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PLAYING WITH ADMISSIBILITY SPECTRA†

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ABSTRACT
Jensen showed that any countable sequence A of A-admissibles is the initial part of the admissibility spectrum of a real R. His construction generalizes straightforwardly to Σ_n-admissibles. This adaptation makes admissibles not in A R-inadmissible. We strengthen Jensen's theorem by requiring that $\Sigma_n(A)$-admissibles not in A be $\Sigma_m(R)$-admissible or $\Sigma_m(R)$-non-projectible, for $m < n$.

§1. Introduction and preliminaries

An ordinal α is *admissible* if $L_\alpha \models \Sigma_1$ Replacement and *A-admissible* ($A \subset \mathrm{ORD}$) if $L_\alpha[A] \models \Sigma_1(A)$ Replacement. Various results connect admissible ordinals with reals. If α is a countable admissible, then there is an $R \subset \omega$ such that α is the first R-admissible greater than ω ($\alpha = \omega_1^R$). (See Barwise [B], Sacks [Sa], and Steel [St] for three different proofs.) There have been different kinds of generalizations of this result. S. Friedman [F1, F2] considers uncountable admissibles. Jensen [J] realizes a countable sequence A of countable A-admissibles as the initial segment of R-admissibles for some real R. Sacks [Sa] finds a solution to $\alpha = \omega_1^R$ minimal in the hyperdegrees, and the author [L] investigates realizing a Jensen-type sequence with minimality at many ordinals. Most of these theorems generalize to Σ_n admissibles (ordinals α such that $L_\alpha \models \Sigma_n$ Replacement).

Another strengthening of Jensen's theorem concerns the ordinals not in A. Using the obvious adaptation of Jensen's proof for realizing a sequence A of $\Sigma_n(A)$-admissibles, the $\Sigma_n(R)$-admisibility of $\alpha \in \mathrm{ORD}/A$ is destroyed by

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sending an ω-sequence through it. This renders it not only $\Sigma_n(R)$-inadmissible but also $\Sigma_1(R)$-inadmissible. Here we show how to clean up spectra while treating the undesirables with gentleness.

The motivation is to kill the Σ_n closure properties of $\alpha \notin A$ while preserving the Σ_m closure, for an arbitrary $m < n$. Phrased this way, the problem becomes more involved, since the intuition of Σ_m closure can be expressed in different ways. Here we consider Σ_m-admissibility and Σ_m-non-projectibility. Our construction breaks into two cases, depending on whether we want to preserve both Σ_m-admissibility and non-projectibility, or just Σ_m-admissibility while destroying Σ_m-non-projectibility.

We conclude with a proof indicating an asymmetry between the two cases, and some questions.

For more detail and proofs regarding the basic material summarized below, see [D] and [MS].

Definitions

$\langle X, \varepsilon \rangle$ is Σ_n-admissible if $\langle X, \varepsilon \rangle \models$ ZF-Power-Replacement + Δ_n Comprehension + Δ_n Bounding + Foundation for definable classes

$\alpha \in$ ORD is Σ_n-admissible if $\langle L_\alpha, \varepsilon \rangle$ is Σ_n-admissible (equivalently, $L_\alpha \models \Sigma_n$ Replacement)

$\alpha \in$ ORD is Σ_n-non-projectible if $L_\alpha \models$ "There is no Σ_n-definable 1–1 function from V into a set"

Σ_n-Adm $= \{\alpha \mid \alpha$ is Σ_n-admissible$\}$

Σ_n-NP $= \{\alpha \mid \alpha$ is Σ_n-non-projectible$\}$

$\hat{\alpha} =$ least primitive recursive closed $\beta > \alpha$

β is $\Sigma_n(\alpha)$-stable if $L_\beta \prec_{\Sigma_n} L_\alpha$

The Σ_n-projectum of α (ρ_n^α) is the least β such that

$$\exists f \in \Sigma_n(L_\alpha) \quad f : \alpha \xrightarrow{1\text{-}1} \beta$$

$A \subset \rho_n^\alpha$ is a $\Sigma_n(L_\alpha)$ Master Code if

$$\forall B \subset \rho_n^\alpha \quad B \in \Delta_{n+1}(L_\alpha) \qquad \text{iff} \quad B \in \Delta_1(L_{\rho_n^\alpha}[A]).$$

If $\mathscr{P}$ is partial order definable over L_α, $G \subseteq \mathscr{P}$ is $\mathscr{P}$-generic (over L_α) if G intersects every dense subset of $\mathscr{P}$ definable over L_α.

PROPOSITION (folklore). (1) *If* $\alpha \in \Sigma_n$-Adm *then* $\alpha \in \Sigma_m$-Adm *and* $\alpha \in \Sigma_m$-NP, $\forall m < n$.

(2) *If* $\alpha \in \Sigma_n$-Adm *and* β *is* $\Sigma_n(\alpha)$-*stable, then* $\beta \in \Sigma_n$-Adm.

(3) *If* $\alpha \in \Sigma_n$-Adm *then* α *is the limit of* α*-many* $\Sigma_{n-1}(\alpha)$*-stables.*

PROPOSITION (folklore). *Suppose* $L_\alpha \models$ *"There is a largest cardinal". Then*
(1) $\alpha \in \Sigma_n$-NP *iff* α *is a limit of* $\Sigma_n(\alpha)$*-stables.*
(2) $\alpha \in \Sigma_n$-Adm *iff* $L_\alpha \models \Delta_n$ *Comprehension.*
$\alpha \in \Sigma_n$-NP *iff* $L_\alpha \models \Sigma_n$ *Comprehension.*

The primary step in proving these is taking the canonical Σ_n Skolem hull of some $\beta < \alpha$ in L_α. The Skolem function is Σ_n, so if α is Σ_n non-projectible then the hull is not all of L_α. Furthermore, if $\beta >$ largest α-cardinal, then the hull is L_γ for some γ. The advantage of assuming a largest cardinal is that the closure properties of admissibility and non-projectibility form a linear hierarchy (Σ_n Adm $< \Sigma_n$ NP $< \Sigma_{n+1}$ Adm), with each ordinal of a class being a limit of ordinals from the next lower class. This is not true in general, as $\aleph_\omega$ is fully non-projectible yet Σ_2 inadmissible. How can we ensure that there always is a largest cardinal?

PROPOSITION (folklore). (1) *Let* $I \subset \alpha$ *be* $\Delta_1(L_\alpha)$. *Let* $\mathscr{P}$ *be the product (with finite support) of the (finite) Levy collapse of each* $\iota \in I$ *to* ω: $p \in \mathscr{P}$ *iff* $\mathrm{dom}(p) \subseteq I$ *is finite and* $p(\iota): n \to \iota$ *is* 1-1, *for some* $n \in \omega$. *Then* $\mathscr{P}$ *preserves* Σ_n *admissibility: if* $\alpha \in \Sigma_n$-Adm *and* G *is* $\mathscr{P}$*-generic, then* α *is* $\Sigma_n(G)$*-Admissible.*

(2) *Let* $I \subseteq$ ORD *be* Δ_1 *uniformly over admissible ordinals. Let* $\mathscr{P}$ *be as above, and* $\mathscr{P}_\alpha$ *be* $(\mathscr{P})^{L_\alpha}$. *Let* $\beta < \alpha$ *and* G_α *be* $\mathscr{P}_\alpha$*-generic. Then* $G_\beta = G_\alpha \cap (\beta \times \omega \times \beta)$ *is* $\mathscr{P}_\beta$*-generic.*

To see part (1), note that $\Vdash$ is definable: if rk $\varphi < \gamma$, φ a bounded formula, then $p \Vdash \varphi$ iff $p \cap (\gamma \times \omega \times \gamma) \Vdash \varphi$ (shown inductively on formulae), so $\Vdash \restriction \mathscr{P} \times \Delta_0$ is Δ_1. If $p \Vdash \varphi$ is a total Σ_n function, then bound rng φ by the following ω-step process. At stage $n+1$, for each $q \in \mathscr{P}_{\alpha_n}$, $x \in \mathrm{dom}\ \varphi$, let $q' \leqq q$ force a value $x(q')$ for $\varphi(x)$. Let α_{n+1} bound the L-ranks of the q's and $x(q')$'s. This construction is bounded by Σ_n admissibility. After ω many steps we have a pre-dense set forcing rng φ into a set.

The second part is true because the components are independent of one another: if $D \subseteq \mathscr{P}_\beta$ is dense, then $\{p \in \mathscr{P}_\alpha \mid (p \restriction \beta \times \omega \times \beta) \in D\}$ is dense in $\mathscr{P}_\alpha$.

All of the preceding definitions, propositions, and proofs relativize to $A \subseteq$ ORD.

§2

THEOREM 1. *Let A be a countable sequence of countable $\Sigma_n(A)$-admissibles. Let $m < n$. There is an $R \subseteq \omega$ such that $\forall \alpha \leqq \sup A$*

(1) $\alpha \in \Sigma_n(R)$-Adm *iff* $\alpha \in A$.

(2) *If* $\alpha \in \Sigma_n(A)$-Adm/A *then* $\alpha \in \Sigma_m(R)$-NP/$\Sigma_{m+1}(R)$-Adm.

(3) *If* $k < n$ *and* $\alpha \in \Sigma_k(A)$-Adm/$\Sigma_n(A)$-Adm *then* $\alpha \in \Sigma_k(R)$-Adm.

PROOF. Without loss of generality we can assume that $\forall \alpha$ ($\leqq \sup A$) $L_{\hat{\alpha}} \models \alpha$ is countable. If this were not the case, let $I = \{\alpha \mid L_{\hat{\alpha}} \models \alpha \text{ is uncountable}\}$, and $\mathscr{P}$ be the Levy collapse of each $\imath \in I \cap \sup A + 1$. With G $\mathscr{P}$-generic, work in $L[G]$ just as we will be working in L. Similarly, we assume $L_\alpha[A] = L_\alpha$ and $\Sigma_k(A)$-Adm $= \Sigma_k$-Adm, since the proof relativizes to $L[A]$.

To each Σ_n admissible α we will associate $D_\alpha \subseteq \alpha$ of ordertype α just as in Jensen's proof, to provide enough scratchwork for each ordinal we're concerned with. The predicate coding $\langle D_\alpha \mid \alpha < \sup A \rangle$ will not affect Σ_n-admissibility or any weaker property. First we describe how to get suitable clubs of Σ_n-inadmissibles. We use these to build the D_α's. The information making α Σ_{m+1}-inadmissible is then coded onto D_α. Finally, the predicate can be coded by a real.

Clubs of Σ_n-inadmissibles

LEMMA 2. *$\forall \alpha \leqq \sup A$ $\exists C_\alpha \subseteq \alpha$ such that*

(1) *C_α is a club of Σ_n-inadmissibles,*

(2) $\forall \beta \leqq \alpha$, $C_\alpha \cap \beta \in L_\beta$,

(3) *if* $\beta < \alpha$, $\beta \in \Sigma_n$-Adm *then* $C_\alpha \cap \beta \in L_\beta$,

(4) *if* $\beta \leqq \alpha$, $\beta \in \Sigma_m$-Adm(-NP) ($m < n$) *then*

$$\beta \in \Sigma_m(C_\alpha)\text{-Adm(-NP)}.$$

PROOF. C_α is any generic for the appropriate forcing $\mathscr{P}_\alpha$ over L_α. Note that by our assumption of countability, such generics exist in $L_{\hat{\alpha}}$.

Inductively on α, let $\mathscr{P}_\alpha$ be $\{ p \mid p$ is a closed set of Σ_n-inadmissibles bounded in α such that (2) and (3) from above hold, and if $\beta \leqq \sup p$ is Σ_m-admissible or -non-projectible then $p \cap \beta$ is bounded in β or $\mathscr{P}_\beta$-generic over $L_\beta\}$. $\leqq$ is end-extension.

Clearly, any $\mathscr{P}_\alpha$-generic in $L_{\hat{\alpha}}$ will satisfy (1)–(3). To show (4), we must show that $\mathscr{P}_\alpha$ preserves Σ_m-admissibility and -non-projectibility.

If φ is Δ_0, let $p \Vdash \varphi$ iff $\sup p > \operatorname{rk} \varphi$ and $L[p] \models \varphi$. Extend $\Vdash$ to Σ_n formulae as usual: $p \Vdash \exists x\, \varphi(x)$ if $\exists x\; p \Vdash \varphi(x)$; for φ unranked, $p \Vdash \neg\varphi$ if $\forall q \leqq p$

$q \nVdash \varphi$. $\Vdash$ is dense in the standard notion of forcing $\Vdash_{st}$: if $p \Vdash_{st} \varphi$ then $\exists q \leqq p$ $q \Vdash \varphi$. So $\Vdash$ suffices for our forcing relation, and is definable: $\Vdash \restriction \mathscr{P} \times \Sigma_n(\Pi_n)$ is $\Sigma_n(\Pi_n)$.

Suppose α is Σ_1-admissible, and $p_{-1} \Vdash \forall n \in \omega\ \exists x_n\ \varphi(n, x_n)$. Let p_i be the least $p \leqq p_{i-1}$ such that that $p \Vdash \varphi(i, t_i)$, for some term t_i of rank $<\sup p_i$. Let $q = \bigcup p_i$. $\sup q$ is inadmissible, as $\langle p_i \mid i \in \omega \rangle$ is $\Delta_1(L_{\sup q})$. Therefore $q' = q \cup \{\sup q\} \leqq p$ and $q' \Vdash \forall n \in \omega\ \exists^{\sup q} x_n\ \varphi(n, x_n)$.

Suppose $\alpha \in \Sigma_m$-NP. Let φ be Σ_m, $p \in \mathscr{P}_\alpha$. We must show Σ_m comprehension holds for $L_\alpha[G_\alpha]$; it suffices to find a $q \leqq p$ such that $\forall n\ q \parallel \varphi(n)$. Let β be the least $\Sigma_m(\alpha)$-stable greater than rk φ. Let $q' \leqq p$ be $\mathscr{P}_\beta$-generic over L_β, $q' \in L_{\hat\beta}$. Let $q = q' \cup \{\beta\}$. q will decide each $\varphi(n)$: $\forall n\ \exists \gamma\ q \cap \gamma \parallel_{L_\beta} \varphi(n)$. Moreover,

$$q \cap \gamma \Vdash_{L_\beta} \varphi(n)\,(\neg\varphi(n)) \qquad \text{iff} \quad q \cap \gamma \Vdash_{L_\alpha} \varphi(n)\,(\neg\varphi(n)),$$

by stability. Since β was the least stable beyond a given ordinal, it's not even Σ_m-non-projectible, much less Σ_n-admissible, so (3) is satisfied.

Suppose $\alpha \in \Sigma_m$-Adm, $m > 1$, φ is Π_{m-1}, and $p \Vdash \forall n \in \omega\ \exists x_n\ \varphi(n, x_n)$. Let $\alpha_0 = \max(\text{rk}\ p, \text{rk(parameters}(\varphi)))$. $\forall q \in \mathscr{P}_\alpha \cap L_{\alpha_0}$, $n \in \omega$ let $q_n \leqq q$, x_n be the least such that $q_n \Vdash \varphi(n, x_n^{q_n})$. This operation has bounded range, say by β_0, by Σ_m-admissibility. Let α_1 be the least $\Sigma_{m-1}(\alpha)$-stable greater than β_0. Continue as before, extending each $q \in \mathscr{P}_\alpha \cap L_{\alpha_1}$ to $q_n \Vdash \varphi(n, x_n^{q_n})$. Let $\alpha_\omega = \bigcup \alpha_n$. As a limit of $\Sigma_{m-1}(\alpha)$-stables, α_ω is also $\Sigma_{m-1}(\alpha)$-stable. Also, for each n

$$D_n = \{q \mid q \Vdash \exists x_n\ \varphi(n, x_n)\}$$

is dense (in L_{α_ω}), by construction. Let $q' \leqq p$, $q' \in L_{\hat\alpha_\omega}$ be $\mathscr{P}_{\alpha_\omega}$-generic. $q = q' \cup \{\alpha_\omega\}$ is a condition, because α_ω is Σ_m-inadmissible (since this construction is $\Delta_m(L_{\alpha_\omega})$). Finally,

$$q \Vdash \forall n \in \omega\ \exists^{\alpha_\omega} x_n\ \varphi(n, x_n). \qquad \square \text{ Lemma 2}$$

The D_α's

Using the C_α's, we can inductively build predicates which assign α-many ordinals to each Σ_n-admissible α.

Let $\gamma_\nu = \sup\{\rho \leqq \nu \mid \rho \in \Sigma_n\text{-Adm}\}$, and

$$\begin{aligned}\theta_\nu = \{ f \in L_{\hat\nu} \mid\ & \text{dom}\ f = \gamma_\nu \\ & \text{rng}\ f = (\gamma_\nu + 1) \cap \Sigma_n\text{-Adm} \\ & \forall \alpha\ f(\alpha) > \alpha \\ & \forall \alpha\ \{\beta\ \beta > \alpha \wedge f^{-1}(\beta) \cap \alpha \neq \varnothing\} \text{ is finite} \\ & \text{letting } F \text{ be } \{\langle \delta, \delta' \rangle \mid f(\delta) = f(\delta')\}\end{aligned}$$

if $\alpha \in \Sigma_m$-Adm $(m \leqq n)$, $L_\alpha[f, F]$ is Σ_m-admissible
if $\alpha \in \Sigma_m$-NP $(m < n)$, $L_\alpha[f, F]$ is Σ_m-np
if $\alpha \in \Sigma_n$-Adm then $f^{-1}(\alpha)$ is unbounded in α
if $\alpha \in \Sigma_n$-Adm and $\beta > \alpha$ then $f^{-1}(\beta) \cap \alpha$ is bounded in $\alpha\}$.

In words, θ_ν consists of those functions (definable near ν) which, to each Σ_n-admissible $\alpha \leqq \nu$, assign α-many smaller ordinals (D_α), keep the D_α's rather separate (last clause), and do not affect admissibility or non-projectibility (up to Σ_n-admissibility).

Lemma 3. *If $\nu' > \nu$ and $f \in \theta_\nu$ then $\exists f' \in \theta_{\nu'} f' \supset f$.*

Proof. By induction on ν', starting with ν.

If $\gamma_{\nu'} < \nu'$, extend f to $f_{\gamma_{\nu'}} \in \theta_{\gamma_{\nu'}}$, and let $f' = f_{\gamma_{\nu'}}$.

If $\gamma_{\nu'} = \nu'$ and $\gamma_{\nu'}$ is a successor Σ_n-admissible (after ι say), let $f_\iota \supseteq f$, $f_\iota \in \theta_\iota$, and let

$$f' = f_\iota \cup \{\langle \alpha, \nu' \rangle \mid \iota \leqq \alpha < \nu'\}.$$

If $\gamma_{\nu'} = \nu'$ is a limit of Σ_n-Adm and is itself Σ_n-inadmissible, let $C_{\nu'} = \langle \nu_j \mid j \leqq \nu' \rangle$ be a club of Σ_n-admissibles from the previous lemma. Let $f_{\nu_{j+1}} \supseteq f_{\nu_j}$, $f_{\nu_{j+1}} \in \theta_{\nu_{j+1}}$ be the least such, and if λ is a limit let $f_{\nu_\lambda} = \bigcup f_{\nu_j}$. $f_{\nu_\lambda} \in \theta_{\nu_\lambda}$ because of the nice properties of $C_{\nu'}$. Let $f' = \bigcup f_{\nu_j}$.

If ν' is a limit of Σ_n-Adm in Σ_n-Adm then we must also ensure that ν' itself gets ν'-many ordinals. Let $f' = \bigcup f_{\nu_j}$ as in the previous case. Let

$$f''(\alpha) = \begin{cases} f'(\alpha) & \alpha \notin C_{\nu'}, \\ \nu' & \alpha \in C_{\nu'}. \end{cases}$$

We use property (3) in the definition of C_α to know that $f''^{-1}(\alpha)$ is unbounded in α, for $\alpha \in \Sigma_n$-Adm. □ Lemma 3

Fix $f \in \theta_{\sup A}$. Let D_α be $f^{-1}(\alpha)$. Let

$$F = \{\langle \delta, \delta' \rangle \mid f(\delta) = f(\delta')\}.$$

Fixing an α

Now we concentrate on how to reduce a given Σ_n-admissible α to a Σ_m-np, Σ_{m+1}-inadmissible. Afterwards we can paste these predicates for different α's together using the D_α's.

Let $g_\alpha \subseteq \alpha$ be the least ω-sequence cofinal in α. We will code g_α into a

Σ_m-generic, Δ_{m+1}-definably. Genericity preserves Σ_m-np; definability uncodes g_α in time to make α Σ_{m+1}-inadmissible.

The forcing involved is Cohen forcing. Let

$$\mathscr{P}^\alpha = \{p \subseteq \gamma < \alpha \mid \forall \delta, p \cap \delta \in L_\delta \text{ and } \delta \in \Sigma_k\text{-Adm(NP)} \text{ iff } \delta \in \Sigma_k(p)\text{-Adm(NP)}\ (k < n)\}.$$

$\Vdash$ is defined as for $\mathscr{P}_\alpha$ from Lemma 2.

$\mathscr{P}^\alpha$ preserves Σ_k-admissibility and -non-projectibility $\forall k$. The proof of this fact is exactly as the proof of the same for $\mathscr{P}_\alpha$.

Let $\{\alpha_i^m \mid i < \alpha\}$ be the club of $\Sigma_m(\alpha)$-stables. Let p_0 be the least $\mathscr{P}^{\alpha_0^m}$-generic over $L_{\alpha_0^m}$. $p_0 \in \mathscr{P}^\alpha$. Let $p_0' = p_0$ if $0 \notin g_\alpha$, $p_0 \cup \{\alpha_0^m\}$ if $0 \in g_\alpha$. More generally, let p_{i+1} be the least $\mathscr{P}^{\alpha_i^m}$-generic through p_i' omitting α_i^m if $i \notin g_\alpha$. Let $p_\lambda = \bigcup p_i$. Let $p_i' = p_i$ if $i \notin g_\alpha$, $p_i \cup \{\alpha_i^m\}$ if $i \in g_\alpha$. Let $p_\alpha = \bigcup p_i$.

If $\alpha_i^m < \beta \leqq \alpha_{i+1}^m$, then β's admissibility or non-projectibility is preserved by p_α, by the definition of $\mathscr{P}^{\alpha_{i+1}^m}$. If $\beta = \alpha_\lambda^m$, λ a limit, β is $\Sigma_m(p_\alpha)$-np: let φ be Σ_m, $\operatorname{rk} \varphi < \alpha_i^m < \alpha_\lambda^m$. p_i will decide each $\varphi(n)$ for $\mathscr{P}^{\alpha_i^m}$. By Σ_m-elementarity, the same decisions are valid for $\mathscr{P}^{\alpha_\lambda^m}$, so in $L_{\alpha_\lambda^m}[p_\alpha]$ φ can be evaluated in $L_{\alpha_i^m}[p_\alpha]$. If $\beta = \alpha_\lambda^m$ is Σ_k-admissible or np $(m < k < n)$, it will be $\Sigma_k(p_\alpha)$-admissible or np : $p_\alpha \restriction \beta$ is $\Delta_{m+1}(L_\beta)$, because $\langle \alpha_i^m \mid i < \lambda \rangle$ is $\Delta_{m+1}(L_{\alpha_\lambda^m})$ and $g_\alpha \restriction \beta$ is finite.

α itself is $\Sigma_m(p_\alpha)$-np, for the same reason that the α_λ^m's remain $\Sigma_m(p_\alpha)$-np. However, α is $\Sigma_{m+1}(p_\alpha)$-inadmissible. $\langle \alpha_i^m \mid i < \alpha \rangle$ is $\Delta_{m+1}(L_\alpha)$, so α can read off g_α from p_α in a Δ_{m+1} way.

The Final Predicate

As a first approximation to the final predicate $B \subseteq \sup A$, spread each p_α along D_α to get p_α'. (If $\alpha \in A$, let $p_\alpha' = \varnothing$.) Let

$$P = \bigcup_{\alpha \in \Sigma_n \mathrm{Adm}} p_\alpha'.$$

Let $\bar{B} = P \oplus f \oplus F$.

Recall that the last two components of $\bar{B}$ do not affect any ordinal's admissibility or non-projectibility. Also, $\alpha \mapsto \bigcup_{\gamma < \alpha} p_\gamma'$ is $\Delta_1(L_\alpha[f, F])$ uniformly in α, since the construction of the previous section is so simply defined. If $\alpha \in \Sigma_n$-Adm, then D_α, recoverable from $\bar{B}$'s third component, is either empty (and $\alpha \in A$) or it brings α down to where it's supposed to be. We must show only that for the finitely many $\beta > \alpha$ such that $D_\beta \cap \alpha \neq \varnothing$, p_β' does not really affect α.

By $\bar{B}$'s third component, $L_\alpha[\bar{B}]$ can separate $\{\gamma < \alpha \mid f(\gamma) > \alpha\}$ into finitely many blocks. Each block separately will preserve α's closure, by the definition of the p_β's. The only possible problem is in the combination of the (finitely many) p'_β's.

This construction must be altered to account for this problem. Instead of defining p_β over L_β, with initial segments preserving the closure of L_γ ($\gamma<\beta$), do it relative to the amount of B constructed thus far. More accurately, assume inductively that for $\gamma<\beta$, p_γ has been defined; furthermore, $\delta\mapsto\bigcup_{\gamma<\delta}p'_\gamma$ is $\Delta_1(L_\delta[f,F])$ uniformly in δ. Let

$$P_\beta=\bigcup_{\gamma<\beta}p'_\gamma,\qquad B_\beta=P_\beta\oplus f\oplus F.$$

Let

$$\mathcal{P}^\beta=\{\,p\subseteq\gamma<\beta\mid\forall\delta,\ p\cap\delta\in L_\delta[B_\beta]\text{ and }\delta\in\Sigma_k(B_\beta)\text{-Adm (-NP)}\ \text{iff}\ \delta\in\Sigma_k(B_\beta,p)\text{-Adm(-NP)}\}.$$

The rest of the constructionof p_β carries through just as before, with everything relativized. This avoids the clash of the p'_β's at α.

Let

$$P=\bigcup_{\alpha\in\Sigma_n\mathrm{Adm}}p'_\alpha.$$

Let $B=P\oplus f\oplus F$.

Finally, B can be coded into a real, by almost disjoint forcing (Jensen [J] or Jensen–Solovay [JS]). If R is the real so produced, and α is closed under addition and $L_\alpha\models V=\mathrm{HC}$, then $B\cap\alpha$ is $\Delta_1(L_\alpha[R])$. Also, admissibility and non-projectibility are not disturbed by R, by its genericity. These properties suffice for the present purpose. □ Theorem 1

§3

Theorem 4. *Let A be a countable sequence of countable $\Sigma_n(A)$-admissibles. Let $m<n$. There is an $R\subseteq\omega$ such that $\forall\alpha\leqq\sup A$*

(1) *$\alpha\in\Sigma_n(R)$-Adm iff $\alpha\in A$.*

(2) *If $\alpha\in\Sigma_n(A)$-Adm/A then $\alpha\in\Sigma_m(R)$-Adm/$\Sigma_m(R)$-NP.*

(3) *If $k<n$ and $\alpha\in\Sigma_k(A)$-Adm/$\Sigma_n(A)$-Adm then $\alpha\in\Sigma_k(R)$-Adm.*

Proof. Most of the machinery of the previous proof carries over, primarily the construction of f and F. All that remains is to define appropriate p_α's.

Even here our previous work is useful. A good technique for reducing α to the appropriate strength is to shoot a club C of Σ_m-inadmissibles through it. If α were $\Sigma_m(C)$-np, let β be $\Sigma_m(C)(\alpha)$-stable. C is unbounded in β, so $\beta\in C$. But

$\beta \in \Sigma_m$-Adm by stability. Therefore, for such a club C, α is not $\Sigma_m(C)$-non-projectible.

The forcing to do this is like the forcing to get a club of Σ_n-inadmissibles. Conditions are closed bounded sequences p of Σ_m-inadmissibles such that (1) $\forall \beta$, $p \cap \beta \in L_{\hat{\beta}}$, (2) if β is Σ_k-non-projectible or -admissible then $p \cap \beta$ is bounded in β or generic for this same forcing over L_β $(k < n)$, and (3) if β is Σ_m-admissible then $p \cap \beta$ is bounded in β. $\leqq$ is end-extension.

Σ_m-admissibility is preserved by the same argument as before. Any stronger admissibility and non-projectibility of any $\beta < \alpha$ is preserved, because the generic is bounded in β, hence is in L_β.

As before, define the p_α's inductively on $\alpha \in \Sigma_n(A)$-Adm, letting p_α be empty if $\alpha \in A$, the least generic over $L_\alpha[\bigcup_{\beta<\alpha} p_\beta \oplus f \oplus F]$ otherwise.

§4

The alert reader will notice that the definitions of the p_α's are fundamentally different in the two proofs. The second proof can be adapted to fit the needs of the first, the appropriate goal being a club of Σ_m-projectibles. Is there an adaptation of the first method to fit the second proof? There are some problems in so doing while retaining the full strength of the theorem.

The p_α's of the first theorem are obtained from a predicate g_α which completely destroys any admissibility, by coding g_α $\Sigma_m \cup \Pi_m$-generically, necessitating a Δ_{m+1} formula to recover it. In adapting this approach to case (2), it seems unlikely that there is a coding (partial $\Sigma_m \cup \Pi_m$ genericity?) which would preserve Σ_m-admissibility but not Σ_m-non-projectibility. In what follows we first find a g_α which makes α a Σ_1-projectible Σ_1-admissible, and code it Δ_m definably. Note how delicate the proof is. Then we state the theorem that this technique yields when plugged into the machinery of Theorem 1. Finally we discuss the limitations of this approach.

Let α be Σ_n-admissible, $n > m$. Assume local countability as in Theorem 1. Let

$$
\begin{aligned}
\mathcal{P}_\alpha = \{f \mid\ & \operatorname{dom} f \subseteq \omega \text{ is finite} \\
& f = f_{\text{fixed}} \cup f_{\text{var}},\ f_{\text{fixed}} \cap f_{\text{var}} = \varnothing \\
& f_{\text{fixed}} \text{ is 1–1 into } \alpha \\
& f_{\text{var}} \text{ is into } \alpha \cup \{\infty\} \text{ (where } \infty \text{ is some arbitrary} \\
& \quad \text{symbol of finite } V\text{-rank)}\}.
\end{aligned}
$$

$$g \leqq f \text{ iff } \operatorname{dom} g \supseteq \operatorname{dom} f$$
$$g_{\text{fixed}} \supseteq f_{\text{fixed}}$$
$$\text{if } n \in \operatorname{dom} f_{\text{var}} \text{ then } g(n) \geqq f(n) \text{ (where } \infty > \alpha).$$

A condition is a partial bijection between ω and α, with commitments that $f(n)$ be at least a certain size, or undefined if $f(n) = \infty$. If G_α is $\mathscr{P}_\alpha$-generic, let g_α be the induced injection from α into ω. Notice that we lose information going from G_α to g_α. "$n \notin \operatorname{rng} g_\alpha$" is $\Delta_1(L_\omega[G_\alpha])$ (viz. $\langle n, \infty\rangle \in G_\alpha$), but only $\Pi_1(L_\alpha[g_\alpha])$.

LEMMA 5. *$L_\alpha[g_\alpha]$ is admissible.*

PROOF. This is a retagging argument, in the style of Steel [St]. The idea is that $p \Vdash \varphi$ depends only on $p \upharpoonright \operatorname{rk} \varphi$, so $\Vdash$ is definable. Henceforth φ is in the language for describing $L_\alpha[g_\alpha]$, not $L_\alpha[G_\alpha]$.

For $f \in \mathscr{P}_\alpha$, $\beta < \alpha$, we define $f \upharpoonright \beta$. $\operatorname{dom}(f \upharpoonright \beta) = \operatorname{dom} f$; and if $f(n) \geqq \beta$ then $f \upharpoonright \beta_{\text{var}}(n) = \beta$, otherwise $f \upharpoonright \beta_{\text{fixed}}(n) = f_{\text{fixed}}(n)$, $f \upharpoonright \beta_{\text{var}}(n) = f_{\text{var}}(n)$. So $f \upharpoonright \beta$ weakens any information above β to β. Let $f \sim_\beta g$ if $f \upharpoonright \beta = g \upharpoonright \beta$.

$\sim_\beta$ satisfies the extension property: If $p_0 \sim_\beta p_1$ and $p_0' \leqq p_0$ then $\exists\, p_1' \leqq p_1$, $p_0' \sim_\beta p_1'$. From this we get the retagging property: If $\operatorname{rk} \varphi < \beta$ and $p_0 \sim_\beta p_1$, then $p_0 \Vdash \varphi$ iff $p_1 \Vdash \varphi$. Retagging is proved by a straightforward induction, using the extension property in the case of negation. Finally, forcing is definable; more exactly, "$p \Vdash \varphi$" is $\Delta_1(L_\beta)$, where $\beta > \operatorname{rk} p, \operatorname{rk} \varphi$, uniformly in β. Again, this is a straightforward induction, except for negation which introduces an unbounded quantifier. In the inductive definition of $\Vdash$ instead of defining "$p \Vdash \neg \varphi$" as "$\forall q \leqq p \;\, q \nVdash \varphi$", let it be "$q \upharpoonright \operatorname{rk} \varphi \nVdash \varphi$".

Using the definability of $\Vdash$ we now show that admissibility is preserved. Suppose $p_0 \Vdash \forall n \in \omega\; \exists x_n\; \varphi(n, x_n)$, $\beta_0 > \operatorname{rk} p_0, \operatorname{rk} \varphi$. To define β_{i+1}, given q, let $\langle q_n, x_n^q\rangle$ be the least such that $q_n \leqq q$, and $q_n \Vdash \varphi(n, x_n^q)$. Let

$$\beta_{i+1} = \sup\{\operatorname{rk}\langle q_n, x_n^q\rangle + 1 \mid q \in \mathscr{P}_\alpha \cap L_{\beta_i},\, n \in \omega\}.$$

Let $\beta = \bigcup_{i \in \omega} \beta_i$. This construction is $\Delta_1(L_\alpha)$, by the definability of $\Vdash$, so $\beta < \alpha$.

Let $i : \mathscr{P}_\beta \hookrightarrow \mathscr{P}_\alpha$ be the identity except that all occurrences of ∞ are replaced by β. Identify $\mathscr{P}_\beta$ with its image. Passing to Boolean completions, $\overline{\mathscr{P}_\beta}$ is a complete sub-algebra of $\overline{\mathscr{P}_\alpha}$, so $G_\beta =_{\text{def}} G_\alpha \cap \mathscr{P}_\beta$ is $\mathscr{P}_\beta$-generic (where G_α is $\mathscr{P}_\alpha$-generic).

$p_0 \Vdash_{\mathscr{P}_\beta} \forall n \in \omega\; \exists x_n\; \varphi(n, x_n)$, by construction, so if $p_0 \in G_\alpha$ then $L_\beta[g_\beta] \models \forall n \in \omega\; \exists x_n\;\; \varphi(n, x_n)$. But $\langle L_\beta[g_\beta], g_\beta\rangle = \langle L_\beta[g_\alpha], g_\alpha\rangle$, so $L_\alpha[g_\alpha] \models \forall n \in \omega\; \exists^\beta x_n\, \varphi(n, x_n)$. □ Lemma 5

Now we code g_α into a $p_\alpha \subseteq \alpha\ \Sigma_{m-1} \cup \Pi_{m-1}$-Cohen generically, Δ_m definably, as in the proof of Theorem 1. Let

$$\mathcal{Q}^\alpha = \{ p \subseteq \gamma < \alpha \mid \forall \delta\ p \cap \delta \in L_\delta[g_\alpha] \text{ and } \delta \in \Sigma_k\text{-Adm(NP)} \\ \text{iff } \delta \in \Sigma_k(p)\text{-Adm(NP) for } k \leqq m\ (k < m)\}.$$

$\leqq$ is end-extension. $\mathcal{Q}^\alpha$ preserves admissibility, just like $\mathcal{P}^\alpha$ from Theorem 1.

Let $\{\alpha_\iota \mid \iota < \alpha\} = \vec{\alpha}$ be the club of $\Sigma_{m-1}(\alpha)$-stables. Start with $p_0 = \varnothing$. Set $p'_\iota = p_\iota \cup \{\alpha_\iota + g_\alpha(\iota)\}$; $p_{\iota+1} = L[g_\alpha]$-least $\mathcal{Q}^{\alpha_{\iota+1}}$-generic through p'_ι over $L_{\alpha_{\iota+1}}[g_\alpha]$; $p_\lambda = \bigcup_{\iota<\lambda} p_\iota$. We must show that each p_ι is a condition, $\vec{\alpha} \restriction \alpha_\iota$ is $\Delta_m(L_{\alpha_\iota})$, and generics over β show up shortly beyond β by local countability so the definability conditions of $\mathcal{Q}^{\alpha_\iota}$ are easily met. We must show that $\mathcal{Q}^{\alpha_\iota}$ preserves admissibility and non-projectibility up to Σ_m-admissiblity.

We need to speak about $\Vdash_{\mathcal{Q}^\alpha}$ in $L[g_\alpha]$. Facts like "$p \Vdash_{\mathcal{Q}^\alpha} \varphi$" are certainly forced by $g \in G_\alpha$; the next lemma shows that g_α suffices for finding such g's.

Lemma 6. *Suppose p, φ are names for a condition and a formula for $\mathcal{Q}^\beta$-forcing ($\beta \leqq \alpha$). Then $\forall g \in \mathcal{P}_\alpha$*

$$g \Vdash \text{"}p \Vdash \varphi\text{"} \quad \textit{iff}\ g \restriction \operatorname{rk} p \Vdash \text{"}p \Vdash \varphi\text{"}.$$

(Recall the convention on $\Vdash_{\mathcal{Q}^\alpha}$, from the $\mathcal{P}_\alpha$ of Theorem 1: for φ ranked, $p \Vdash \varphi$ iff $\sup p > \operatorname{rk} \varphi$ and $L[p] \models \varphi$.)

Proof. $g \leqq g \restriction \operatorname{rk} p$, so $\Leftarrow$ is trivial.

Suppose $g \Vdash$ "$p \Vdash \varphi$". If φ is bounded and $p \Vdash \varphi$, then $\operatorname{rk} p > \operatorname{rk} \varphi$, so rk "$p \Vdash \varphi$" $= \operatorname{rk} p$. By the retagging lemma, $g \restriction \operatorname{rk}(p) \Vdash$ "$p \Vdash \varphi$".

If $\varphi = \exists x\, \varphi'(x)$, let τ be such that $g \Vdash$ "$p \Vdash \varphi'(\tau)$". Inductively, $g \restriction \operatorname{rk} p \Vdash$ "$p \Vdash \varphi'(\tau)$".

If $\varphi = \forall x\, \varphi'(x)$, let $f \leqq g \restriction \operatorname{rk} p$. Let $f' \leqq f$, $g' \leqq g$ be such that

dom $f' =$ dom g', $f' \restriction \operatorname{rk} p = g' \restriction \operatorname{rk} p$,
$n \in \operatorname{dom} f'/\operatorname{dom} f \Rightarrow f'_{\text{var}}(n) = \operatorname{rk} p$,
$n \in \operatorname{dom} g'/\operatorname{dom} g \Rightarrow g'_{\text{var}}(n) = \operatorname{rk} p$, and
each of dom f', dom g' is sufficiently larger than dom f, dom g.

Let $\theta: \omega \to \omega$ be a permutation such that
(1) $\theta = \text{Id}$ off of dom g',
(2) $g'_{\text{fixed}}(n) \overset{=}{\neq} f'_{\text{fixed}}(m) \Rightarrow \theta(n) \overset{=}{\neq} m$,
(3) $g'_{\text{fixed}}(n) < f'_{\text{var}}(m) \Rightarrow \theta(n) \neq m$,
(4) $f'_{\text{fixed}}(m) < g'_{\text{var}}(n) \Rightarrow \theta(n) \neq m$.

θ induces an automorphism of $\mathscr{P}_\beta: \theta(p)(n) = p(\theta^{-1}(n))$. By construction, f' and $\theta(g)$ are compatible; let $f'' \leqq f', \theta(g')$.

$$f'' \Vdash \theta(\text{“}p \Vdash \varphi\text{”}) = \text{“}\theta(p) \Vdash_{\theta(\mathscr{Q})} \theta(\varphi)\text{”}.$$
$$f'' \Vdash \theta(p) = p \text{ because if } \theta(n) \neq n \text{ then } f'' \Vdash g_\beta(n) \geqq \text{rk } p.$$

Also, $f'' \Vdash \theta(\varphi) = \varphi$ since φ is a formula in the language for $L_\beta[p_\beta]$. Finally, $\theta(\mathscr{Q}^\beta) = \mathscr{Q}^\beta$ because $\mathscr{Q}^\beta$ uses G^β in its definition only insofar as it considers members of $L_\delta[g_\beta]$, which are unchanged by the finite permutation θ. So $f'' \Vdash$ “$p \Vdash \varphi$”. Therefore $g \upharpoonright \text{rk } p \Vdash \neg\neg$“$p \Vdash \varphi$”. Since φ begins with $\forall$, $g \upharpoonright \text{rk } p \Vdash$ “$p \Vdash \varphi$”. □ Lemma 6

LEMMA 7. *For* $\beta = \alpha_{\iota+1}$ (*so* $\beta \in \Sigma_{m-1}$-Adm/Σ_{m-1}-NP), $\beta \in \Sigma_{m-1}(p_\beta)$-Adm.

PROOF. Let MC be the Σ_{m-2} master code for L_β. $L_\beta[\text{MC}]$ is admissible. By Lemma 5, $L_\beta[\text{MC}, g_\beta]$ is admissible. By the genericity of p_β, $L_\beta[\text{MC}, g_\beta, p_\beta]$ is admissible.

Let p, φ range over g_β-names for $\mathscr{Q}^\beta$ conditions and $\Sigma_{m-2} \cup \Pi_{m-2}(L_\beta[p_\beta])$ formulae. p^{g_β}, φ^{g_β} are their realizations in $L_\beta[g_\beta]$. Let

$$\text{MC}'_{\mathscr{Q}} = \{\langle g, p, \varphi\rangle \mid g = g \upharpoonright \text{rk } p \wedge g \Vdash \text{“}p \Vdash \varphi\text{”}\}.$$

MC'_{p_β} is $\Delta_{m-1}(L_\beta)$, hence $\Delta_1(L_\beta[\text{MC}])$. In $L_\beta[\text{MC}, g_\beta]$, let

$$\text{MC}_{\mathscr{Q}} = \{\langle p, \varphi\rangle \mid \exists g \text{ compatible with } g_\beta, \langle g, p, \varphi\rangle \in \text{MC}'_{\mathscr{Q}}\}.$$

$\text{MC}_{\mathscr{Q}}$ is $\Delta_1(L_\beta[\text{MC}, g_\beta])$, so $L_\beta[\text{MC}_{\mathscr{Q}}, p_\beta]$ is admissible.

By Lemma 6, $\text{MC}_{\mathscr{Q}}$ determines $\Vdash_{\mathscr{Q}} \upharpoonright \Sigma_{m-2} \cup \Pi_{m-2}$. Let

$$\text{MC}_{p_\beta} = \{\varphi \mid \exists p \in p_\beta \langle p, \varphi\rangle \in \text{MC}_{\mathscr{Q}}\}.$$

MC_{p_β} is $\Delta_1(L_\beta[\text{MC}_{\mathscr{Q}}, g_\beta])$, so $L_\beta[\text{MC}_{p_\beta}, p_\beta]$ is admissible. Furthermore, MC_{p_β} is the $\Sigma_{m-2}(L_\beta[p_\beta])$-master code. Therefore, $L_\beta[p_\beta]$ is Σ_{m-1}-admissible.

□ Lemma 7

LEMMA 8. *For* $\beta = \alpha_\lambda$, λ *a limit* (*so* $\beta \in \Sigma_{m-1}$-NP), $\beta \in \Sigma_{m-1}(p_\beta)$-NP.

PROOF. Extend the rank function to all formulae, by setting $\text{rk}(\exists x\, \varphi) = \text{rk } \varphi$. Let φ be $\Sigma_{m-1}(L_\beta[p_\beta])$, $\gamma > \text{rk } \varphi$ a successor $\Sigma_{m-1}(\beta)$-stable. Let $\text{MC}'_{\mathscr{Q}^\gamma}$, $\text{MC}_{\mathscr{Q}^\gamma}$, MC_{p_γ} be as in Lemma 7, only also allowing $\Sigma_{m-1} \cup \Pi_{m-1}$ formulae. By Lemma 7, there are $g \in G_\gamma$, $p \in p_\gamma$, $\langle g, p, \varphi\rangle \in \text{MC}'_{\mathscr{Q}^\gamma}$ or $\langle g, p, \neg\varphi\rangle \in \text{MC}'_{\mathscr{Q}^\gamma}$. $g = g \upharpoonright \text{rk } p < \gamma$, so $g \in G_\beta$, $p \in p_\beta$. By stability, $\langle g, p, (\neg)\varphi\rangle \in \text{MC}'_{\mathscr{Q}^\beta}$. So Σ_{m-1} truth in $L_\beta[p_\beta]$ reflects. □ Lemma 8

LEMMA 9. *If* $\beta = \alpha_\lambda \in \Sigma_m$-Adm *then* $\beta \in \Sigma_m(p_\beta)$-Adm.

PROOF. Let $\mathrm{MC}'_{\mathscr{Y}}$, $\gamma \leqq \beta$, be as in Lemma 8. We have already seen that for $\varphi \in \Sigma_{m-1} \cup \Pi_{m-1}$, g_γ and p_γ determine φ in both $L_\gamma[p_\gamma]$ and $L_\beta[p_\beta]$, for γ the least $\Sigma_{m-1}(\beta)$-stable $>$ rk φ. So $\{\varphi \mid \exists \gamma > \mathrm{rk}\,\varphi, \gamma\ \Sigma_{m-1}(\beta)\text{-stable}, \exists g \text{ compatible with } g_\beta \text{ and } p \text{ with } p_\beta \langle g, p, \varphi\rangle \in \mathrm{MC}'_{\mathscr{Y}}\} = \mathrm{MC}_{p_\beta}$ is the $\Sigma_{m-1}(L_\beta[p_\beta])$-master code. The sequence of stables, as well as the $\mathrm{MC}'_{\mathscr{Y}}$'s, are $\Delta_m(L_\beta)$, hence $\Delta_1(L_\beta[\mathrm{MC}])$, where MC is the $\Sigma_{m-1}(L_\beta)$-master code. $L_\beta[\mathrm{MC}]$ is admissible, so $L_\beta[\mathrm{MC}, g_\beta, p_\beta]$ is also, as well as $L_\beta[\mathrm{MC}_{p_\beta}, p_\beta]$. So $L_\beta[p_\beta]$ is Δ_m-admissible.

□ Lemma 9

Since $\vec{\alpha}$ is $\Delta_m(L_\alpha)$ and length $\vec{\alpha} = \alpha$, g_α is $\Delta_m(L_\alpha[p_\alpha])$, so $L_\alpha[p_\alpha]$ is Σ_m-projectible. Using these p_α's, we get the following:

THEOREM 10. *Let A be a countable sequence of countable $\Sigma_n(A)$-admissibles. Let $m < n$. There is an $R \subseteq \omega$ such that $\forall \alpha \leqq \sup A$*

(1) $\alpha \in \Sigma_n(R)$-Adm *iff* $\alpha \in A$.

(2) *If* $\alpha \in \Sigma_n(A)$-Adm/A *then* $\alpha \in \Sigma_m(R)$-Adm/$\Sigma_m(R)$-NP.

(3) *If* $k \leqq m$ *and* $\alpha \in \Sigma_k(A)$-Adm *then* $\alpha \in \Sigma_k(R)$-Adm.

(3) holds because all properties strictly weaker then Σ_m-non-projectibility were preserved everwhere at all times. (2) holds by the construction of p_α. (1) holds primarily because for each of the finitely many $\beta > \alpha$ such that $D_\beta \cap \alpha \neq \varnothing$, $D_\beta \cap \alpha$ is bounded in α. So when α recovers g_β(as much as it can) in a Δ_m way, all it gets is $g_\beta \cap (\gamma \times \omega)$, some $\gamma < \alpha$, which is set-generic over L_α. Set genericity does not impair any closure. The failure of this theorem is that we have no such guarantee for $\alpha \in \Sigma_k(A)$-Adm/$\Sigma_n(A)$-Adm, where $m < k < n$. There could be a $\beta > \alpha$ such that $L_\alpha \prec_{\Sigma_{m-1}} L_\beta$ and order-type $(D_\beta \cap \alpha) = \alpha$. In this case, $g_\beta \cap (\alpha \times \omega)$ is $\Delta_m(L_\alpha[p_\beta \cap \alpha])$, and is an injection of α into ω.

The point of this is to sharpen our understanding of these closure properties. Given local countability they fall into a neat linear order based on Comprehension:

$$\Delta_1 \text{ Comp} < \Sigma_1 \text{ Comp} < \Delta_2 \text{ Comp} < \Sigma_2 \text{ Comp} < \cdots.$$

From this picture, one might conjecture homogeneity, in that the passage from one point in this sequence to the next should be just like the passage from any other point to its successor. Our work suggests that this is false, and that Σ_m-admissibles and -non-projectibles are more closely tied than are Σ_m-non-projectibles and Σ_{m+1}-admissibles. Can this be made precise? Is there a forcing partial order over which a certain partial genericity preserves Σ_m-admissiblity but not Σ_m-non-projectibility?

Along a different line, one can ask about starting with a sequence $A \subseteq \Sigma_n(A)$-NP.† This paper's construction relied on local countability, for the existence of generics. Levy forcing does not always preserve non-projectibility: if $L_\alpha \models V = L_{\aleph_\omega}$ and $L_\alpha[g] \models V = \mathrm{HC}$, then $L_\alpha[g] \not\Vdash \Sigma_2$ NP. So if such an α is the first member of A, then A cannot be realized as the Σ_2-NP spectrum of a real. If we restrict our attention to those sequences which seem not to demand local countability, then S. Friedman has shown [F], [F2] that even for the first case, getting α to be the least R-admissible $> |\alpha|$ for some $R \subseteq |\alpha|$, not all admissibles have such an R. So for a construction without cardinal collapses, or for retaining a sequence of Σ_n-non-projectibles, what are good analogues of the present theorems?

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